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Convergence of Generalized C-Fractions

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INTRODUCTION

An important part of the study of the analytic behavior of continued fractions (Perron [7], Wall [15]) is devoted to so called C-fractions corresponding to a formal power series. These C-fractions also appear, with a minor modification, in the first and second coefficients, as continued fractions of which the approximants are nothing but the approximants on a normal stepline taken from the Padé table for a certain power series (the corresponding power series if we consider the main stepline).

In view of this connection it is possible to derive convergence of sequences of Padé approximants, using the behavior of continued fractions. Thus following theorem due to van Vleck [14] and Pringsheim [8] (see also Perron [7, p. 148]) is of some interest.

THEOREM. *Consider a nonterminating regular C-fraction*

$$1 + \frac{a_1 x|}{|1} + \frac{a_2 x|}{|1} + \cdots + \frac{a_\nu x|}{|1} + \cdots \qquad (a_\nu \neq 0 \text{ for } \nu \in \mathbb{N}). \qquad (0.1)$$

A. *Let the coefficients satisfy* $|a_\nu| \leqslant g$ $(\nu \in \mathbb{N})$. *Then the C-fraction converges to an analytic nonrational function* $F(x)$ *on* $\mathfrak{D} = \{x \in \mathbb{C} \mid |x| < 1/4g\}$; *on* $\mathfrak{D}$ *this function* $F(x)$ *equals the corresponding (formal) power series of* (0.1).

B. *Let the coefficients satisfy* $\limsup_{\nu\to\infty} |a_\nu| \leqslant g$. *Then the C-fraction converges to a meromorphic nonrational function on* $\mathfrak{D}$ *(as in A); in the poles of the limit function* (0.1) *is inessentially divergent. On a neighborhood of* $x = 0$ *the limit function equals the corresponding (formal) power series of* (0.1).

Remark. It might be convenient to recall some facts about continued fractions.

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(a) If

$$\eta_\nu(x) = 1 + \frac{a_1 x|}{|1} + \frac{a_2 x|}{|1} + \cdots + \frac{a_\nu x|}{|1} \qquad (\nu \in \mathbb{N}), \quad \eta_0 = 1,$$

then $\eta_\nu = A_\nu / B_\nu$ ($\nu \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$), where the A's and B's satisfy the recurrence relation $\Omega_\mu = \Omega_{\mu-1} + a_\mu x \Omega_{\mu-2}$ ($\mu = 1,..., \nu$) with initial values $A_{-1} = A_0 = B_0 = 1$, $B_{-1} = 0$.

(b) The A's and B's are polynomials in x with $A_\nu(0) = B_\nu(0) = 1$ ($\nu \in \mathbb{N}_o$).

(c) The continued fraction (0.1) is called *convergent* to a function $f(x)$ on $\mathfrak{D} \subset \mathbb{C}$ if $\lim_{\nu\to\infty} \eta_\nu(x) = f(x)$ for all $x \in \mathfrak{D}$; uniform convergence on $\mathfrak{D}$ is defined in the usual way.

(d) The continued fraction (0.1) is called *inessentially divergent* in x_0 if $\lim_{\nu\to\infty} B_\nu(x_0)/A_\nu(x_0) = 0$.

Because in the study of a certain generalization of the Padé table (to a simultaneous Padé table for n formal power series) the generalized steplines give rise to a kind of generalized C-fraction (see de Bruin [2]) it is natural to study the analytic behavior of these multidimensional continued fractions to derive convergence results for the simultaneous Padé table (see de Bruin [3]).

The generalization of the notion of continued fraction suitable for our purposes can be seen as a form of the so-called *Jacobi–Perron algorithm* (Perron [4]); an algorithm that has been studied from different viewpoints by o.a. Bernstein [1] and Schweiger [9].

In Section 1 of this paper the generalization of a continued fraction is given, along with a very elegant definition made possible by the extension of a method described by Thron [11, 12] for the ordinary case.

Then follows the definition of the concept of convergence (Section 2), the generalization of a C-fraction including the correspondence with (in this situation) n-tuples of formal power series (Section 3) and the notion of a regular C-n-fraction (Section 4).

In Section 5 the main results on convergence are given, which then are proved by a chain of lemmas in Section 6. The method of proof employed, is a direct generalization of the method of proof for the ordinary C-fraction as given in Perron [7, pp. 64, 148].

It must be noted, however, that the convergence theorems are by no means best possible in the sense that they imply convergence of the regular C-n-fraction on the entire disc $\{x \in \mathbb{C} \mid |x| < \min(R_1, ..., R_n)\}$, where R_j is the radius of convergence of the jth corresponding power series ($j = 1,..., n$); see Example 5.1.

1. n-FRACTIONS

Let n be a fixed natural number and let the quantities $b_0^{(i)}$, $a_\nu^{(i)}$, b_ν ($i = 1,\ldots, n$; $\nu \in \mathbb{N}$) be (as yet unknown) complex numbers, $b_\nu \neq 0$ ($\nu \in \mathbb{N}$), which may depend on a (some) variable(s).

An *n-fraction* is now a set of n-fractions, with common denominator, of the form given below,

$$b_0^{(1)} + \cfrac{a_1^{(1)}}{b_1 + \cfrac{a_2^{(n)} + \cfrac{a_3^{(n-1)}+\cdots}{b_3+\cdots}}{b_2 + \cfrac{a_3^{(n)}+\cdots}{b_3+\cdots}}}, \qquad b_0^{(2)} + \cfrac{a_1^{(2)} + \cfrac{a_2^{(1)}}{b_2 + \cfrac{a_3^{(n)}+\cdots}{b_3+\cdots}}}{b_1 + \cfrac{a_2^{(n)}+\cfrac{a_3^{(n-1)}+\cdots}{b_3+\cdots}}{b_2+\cfrac{a_3^{(n)}+\cdots}{b_3+\cdots}}}, \ldots,$$

$$b_0^{(n)} + \cfrac{a_1^{(n)} + \cfrac{a_2^{(n-1)} + \cfrac{a_3^{(n-2)}+\cdots}{b_3+\cdots}}{b_2+\cfrac{a_3^{(n)}+\cdots}{b_3+\cdots}}}{b_1+\cfrac{a_2^{(n)}+\cfrac{a_3^{(n-1)}+\cdots}{b_3+\cdots}}{b_2+\cfrac{a_3^{(n)}+\cdots}{b_3+\cdots}}}.$$

The manner in which the successive fractions are formed, can easily be deduced from the following diagram:

$$\begin{array}{lcl|cl|ccl|cl|c}
 & & a_1^{(1)} & & a_2^{(1)} & & \cdots & a_{\nu-1}^{(1)} & & a_\nu^{(1)} & \cdots \\
b_0^{(1)} & + & a_1^{(2)} & + & a_2^{(2)} & + & \cdots + & a_{\nu-1}^{(2)} & + & a_\nu^{(2)} & + \cdots \\
\vdots & & \vdots & & \vdots & & & \vdots & & \vdots & \\
b_0^{(n-2)} & + & a_1^{(n-1)} & + & a_2^{(n-1)} & + & \cdots + & a_{\nu-1}^{(n-1)} & + & a_\nu^{(n-1)} & + \cdots \\
b_0^{(n-1)} & + & a_1^{(n)} & + & a_2^{(n)} & + & \cdots + & a_{\nu-1}^{(n)} & + & a_\nu^{(n)} & + \cdots \\
\hline
b_0^{(n)} & + & b_1 & + & b_2 & + & \cdots + & b_{\nu-1} & + & b_\nu & + \cdots .
\end{array} \tag{1.1}$$

Terminating the n-fraction after the 1st, 2nd,..., column, we get the so called *approximant n-tuples*

$$b_0^{(1)}, b_0^{(2)}, \ldots, b_0^{(n)},$$

$$b_0^{(1)} + \frac{a_1^{(1)}}{b_1}, b_0^{(2)} + \frac{a_1^{(2)}}{b_1}, \ldots, b_0^{(n)} + \frac{a_1^{(n)}}{b_1},$$

$$b_0^{(1)} + \cfrac{a_1^{(1)}}{b_1 + \cfrac{a_2^{(n)}}{b_2}}, b_0^{(2)} + \cfrac{a_1^{(2)} + \cfrac{a_2^{(1)}}{b_2}}{b_1 + \cfrac{a_2^{(n)}}{b_2}}, \ldots, b_0^{(n)} + \cfrac{a_1^{(n)} + \cfrac{a_2^{(n-1)}}{b_2}}{b_1 + \cfrac{a_2^{(n)}}{b_2}}.$$
$$\vdots$$

From this we see that adding a column to the n-fraction (i.e., $\nu - 1 \to \nu$) results in replacing $a_{\nu-1}^{(1)}, a_{\nu-1}^{(2)}, \ldots, a_{\nu-1}^{(j)}, \ldots, a_{\nu-1}^{(n)}, b_{\nu-1}$ by $a_{\nu-1}^{(1)}, a_{\nu-1}^{(2)} + a_\nu^{(1)}/b_\nu, \ldots, a_{\nu-1}^{(j)} + a_\nu^{(j-1)}/b_\nu, \ldots, a_{\nu-1}^{(n)} + a_\nu^{(n-1)}/b_\nu$, and $b_{\nu-1} + a_\nu^{(n)}/b_\nu$, respectively. The numerator is found one row higher up and the denominator (always b_ν) in the last row of diagram (1.1).

In the sequel we use for a nonterminating n-fraction the notation

$$\begin{pmatrix} & a_1^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ b_0^{(1)} & \vdots & & \vdots & \\ \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_0^{(n)} & b_1 & \cdots & b_\nu & \cdots \end{pmatrix} \tag{1.2}$$

and for an n-fraction terminating with column with index ν and final approximant n-tuple $\xi_0^{(1)}, \ldots, \xi_0^{(n)}$ the notation

$$\begin{pmatrix} \xi_0^{(1)} \\ \vdots \\ \xi_0^{(n)} \end{pmatrix} \stackrel{k}{=} \begin{pmatrix} & a_1^{(1)} & \cdots & a_\nu^{(1)} \\ b_0^{(1)} & \vdots & & \\ \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} \\ b_0^{(n)} & b_1 & \cdots & b_\nu \end{pmatrix}. \tag{1.3}$$

If we take $n = 1$ (and omit the part between the dotted lines in diagram (1.1)), we are led to the definition and notation for an ordinary continued fraction as given in, for instance, Perron [6, 7].

As in the references just mentioned, an n-fraction (1.3) can be calculated "backwards." Put

$$\begin{pmatrix} \xi_\nu^{(1)} \\ \vdots \\ \xi_\nu^{(n)} \end{pmatrix} = \begin{pmatrix} a_\nu^{(2)} \\ \vdots \\ a_\nu^{(n)} \\ b_\nu \end{pmatrix}, \qquad \begin{pmatrix} \xi_k^{(1)} \\ \vdots \\ \xi_k^{(n)} \end{pmatrix} \stackrel{k}{=} \begin{pmatrix} & a_{k+1}^{(1)} & \cdots & a_\nu^{(1)} \\ a_k^{(2)} & \vdots & & \vdots \\ \vdots & & & \\ a_k^{(n)} & a_{k+1}^{(n)} & \cdots & a_\nu^{(n)} \\ b_k & b_{k+1} & \cdots & b_\nu \end{pmatrix}$$
$$(k = 1, \ldots, \nu - 1).$$

Then (1.1) with last column with index ν leads to

$$
\begin{aligned}
&\xi_{\nu-1}^{(1)} = a_{\nu-1}^{(2)} + \frac{a_\nu^{(1)}}{\xi_\nu^{(n)}}, \quad \xi_{\nu-2}^{(1)} = a_{\nu-2}^{(2)} + \frac{a_{\nu-1}^{(1)}}{\xi_{\nu-1}^{(n)}}, \dots, \xi_1^{(1)} = a_1^{(2)} + \frac{a_2^{(1)}}{\xi_2^{(n)}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \xi_0^{(1)} = b_0^{(1)} + \frac{a_1^{(1)}}{\xi_1^{(n)}}, \\
&\xi_{\nu-1}^{(2)} = a_{\nu-1}^{(3)} + \frac{\xi_\nu^{(1)}}{\xi_\nu^{(n)}}, \quad \xi_{\nu-2}^{(2)} = a_{\nu-2}^{(3)} + \frac{\xi_{\nu-1}^{(1)}}{\xi_{\nu-1}^{(n)}}, \dots, \xi_1^{(2)} = a_1^{(3)} + \frac{\xi_2^{(1)}}{\xi_2^{(n)}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \xi_0^{(2)} = b_0^{(2)} + \frac{\xi_1^{(1)}}{\xi_1^{(n)}}, \\
&\vdots \\
&\xi_{\nu-1}^{(n-1)} = a_{\nu-1}^{(n)} + \frac{\xi_\nu^{(n-2)}}{\xi_\nu^{(n)}}, \xi_{\nu-2}^{(n-1)} = a_{\nu-2}^{(n)} + \frac{\xi_{\nu-1}^{(n-2)}}{\xi_{\nu-1}^{(n)}}, \dots, \xi_1^{(n-1)} = a_1^{(n)} + \frac{\xi_2^{(n-2)}}{\xi_2^{(n)}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \xi_0^{(n-1)} = b_0^{(n-1)} + \frac{\xi_1^{(n-2)}}{\xi_1^{(n)}}, \\
&\xi_{\nu-1}^{(n)} = b_{\nu-1} + \frac{\xi_\nu^{(n-1)}}{\xi_\nu^{(n)}}, \quad \xi_{\nu-2}^{(n)} = b_{\nu-2} + \frac{\xi_{\nu-1}^{(n-1)}}{\xi_{\nu-1}^{(n)}}, \dots, \xi_1^{(n)} = b_1 + \frac{\xi_2^{(n-1)}}{\xi_2^{(n)}}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \xi_0^{(n)} = b_0^{(n)} + \frac{\xi_1^{(n-1)}}{\xi_1^{(n)}}.
\end{aligned}
\tag{1.4}
$$

It is also possible to use the Euclidean greatest common divisor algorithm for an n-tuple of integers which leads to an infinite number of linear equations in an infinite number of unknowns; then it is easy to derive the recurrence relations for the numerators and denominators of the sequence of approximant n-tuples given in Theorem 1.1 (Perron [4], de Bruin [2]).

THEOREM 1.1. *Consider an n-fraction of form* (1.2) *and denote the approximant n-tuples by* $(\{\eta_\nu^{(i)}\}_{i=1}^n)_{\nu=0}^\infty$ (i.e., *the values of the terminated n-fractions* (1.3) *connected with* (1.2)). *Then* $\eta_\nu^{(i)} = A_\nu^{(i)}/B_\nu$ $(i = 1,\dots, n;\ \nu \in \mathbb{N}_0)$ *in which the numerators and denominators satisfy a recurrence relation with initial values as given below,*

$$
\begin{aligned}
&\Omega_\nu = b_\nu\Omega_{\nu-1} + a_\nu^{(n)}\Omega_{\nu-2} + a_\nu^{(n-1)}\Omega_{\nu-3} + \cdots + a_\nu^{(1)}\Omega_{\nu-n-1} \\
&\qquad\qquad\qquad (\nu \in \mathbb{N};\ \text{for } \Omega \text{ read } A^{(1)},\dots, A^{(n)}, B), \\
&A_{-j}^{(i)} = \delta_{i+j,n+1}, \qquad B_{-j} = 0\ (i, j = 1,\dots, n), \\
&A_0^{(i)} = b_0^{(i)} \qquad\quad (i = 1,\dots, n), \quad B_0 = 1.
\end{aligned}
\tag{1.5}
$$

COROLLARY 1.1. *For an n-fraction* (1.3) *with last column* $(a_\nu^{(1)}, \xi_\nu^{(1)},..., \xi_\nu^{(n)})$, *we have*

$$\xi_0^{(i)} = \frac{\xi_\nu^{(n)} A_{\nu-1}^{(i)} + \xi_\nu^{(n-1)} A_{\nu-2}^{(i)} + \cdots + \xi_\nu^{(1)} A_{\nu-n}^{(i)} + a_\nu^{(1)} A_{\nu-n-1}^{(i)}}{\xi_\nu^{(n)} B_{\nu-1} + \xi_\nu^{(n-1)} B_{\nu-2} + \cdots + \xi_\nu^{(1)} B_{\nu-n} + a_\nu^{(1)} B_{\nu-n-1}} \qquad (i = 1,..., n),$$

where the A's and B's follow from (1.5).

Remark 1.1. For $n = 1$ we have again the formulas for an ordinary continued fraction.

There is, however, a far more elegant way of defining an n-fraction if one uses a generalization of the method of successive linear fractional transformations (which is extremely useful in deriving convergence results) as described by Thron [12] (for more details see [11]) for an ordinary continued fraction.

Consider an n-fraction (1.2) with $b_0^{(i)} = 0$ $(i = 1,..., n)$ and construct the following transformations $s_\nu^{(i)} : \mathbb{C}^n \to \mathbb{C}$ $(i = 1,..., n;\ \nu \in \mathbb{N})$

$$s_\nu^{(1)}(x_1,..., x_n) = \frac{a_\nu^{(1)}}{b_\nu + x_n}, \qquad s_\nu^{(i)}(x_1,..., x_n) = \frac{a_\nu^{(i)} + x_{i-1}}{b_\nu + x_n} \qquad (i = 2,..., n). \tag{1.6}$$

Using (1.6) we define another sequence of transformation n-tuples in an inductive way

$$S_\nu^{(i)}(x_1,..., x_n) = S_{\nu-1}^{(i)}(s_\nu^{(1)},..., s_\nu^{(n)}) \qquad (i = 1,..., n;\ \nu \in \mathbb{N}\backslash\{1\}) \tag{1.7}$$

with initial n-tuple

$$S_1^{(i)}(x_1,..., x_n) = s_1^{(i)}(x_1,..., x_n) \qquad (i = 1,..., n). \tag{1.8}$$

THEOREM 1.2. *The transformations* (1.7) *and* (1.8) *satisfy*

$$S_\nu^{(i)}(x_1,..., x_n) = \frac{A_\nu^{(i)} + x_n A_{\nu-1}^{(i)} + x_{n-1} A_{\nu-2}^{(i)} + \cdots + x_1 A_{\nu-n}^{(i)}}{B_\nu + x_n B_{\nu-1} + x_{n-1} B_{\nu-2} + \cdots + x_1 B_{\nu-n}}$$
$$(i = 1,..., n;\ \nu \in \mathbb{N}), \tag{1.9}$$

where the A's and B's follow from (1.5) *with* $b_0^{(i)} = 0$ $(i = 1,..., n)$.

Proof. Follows by induction on ν. ∎

It is now possible to define the concept of an n-fraction in the following way

DEFINITION 1.1. Let $b_0^{(i)}, a_\nu^{(i)}, b_\nu$ $(i = 1,..., n;\ \nu \in \mathbb{N})$ be complex numbers and define the transformations $S_\nu^{(i)} : \mathbb{C}^n \to \mathbb{C}$ $(i = 1,..., n;\ \nu \in \mathbb{N})$ as in (1.7) and (1.8); $S_0^{(i)} \equiv 0$ $(i = 1,..., n)$. Then the sequence of n-tuples $(\{b_0^{(i)} + S_\nu^{(i)}(0,..., 0)\}_{i=1}^n)_{\nu=0}^\infty$ is called an *n-fraction*.

As for the ordinary continued fraction we can give an n-fraction by means of a product of matrices.

Let $A_\nu^{(1)},\dots, A_\nu^{(n)}, B_\nu$ be the numerators and denominator of an n-fraction (1.2) as given by Theorem 1.1 and define

$$\mathfrak{A}_\nu = \begin{pmatrix} A_\nu^{(n)} & A_{\nu-1}^{(n)} & \cdots & A_{\nu-n}^{(n)} \\ \vdots & \vdots & & \vdots \\ A_\nu^{(1)} & A_{\nu-1}^{(1)} & \cdots & A_{\nu-n}^{(1)} \\ B_\nu & B_{\nu-1} & \cdots & B_{\nu-n} \end{pmatrix} \quad (\nu \in \mathbb{N}), \qquad \mathfrak{A}_0 = \begin{pmatrix} b_0^{(n)} & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_0^{(1)} & \vdots & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$\mathfrak{B}_\nu = \begin{pmatrix} b_\nu & 1 & 0 & \cdots & 0 \\ a_\nu^{(n)} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_\nu^{(1)} & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad (\nu \in \mathbb{N}).$$

Then it is obvious that $\mathfrak{A}_\nu = \mathfrak{A}_{\nu-1}\mathfrak{B}_\nu$ ($\nu \in \mathbb{N}$), from which we derive for later use

$$\det \mathfrak{A}_0 = (-1)^n, \qquad \det \mathfrak{A}_\nu = (-1)^{n(\nu+1)}\, a_1^{(1)} a_2^{(1)} \cdots a_\nu^{(1)} \qquad (\nu \in \mathbb{N}). \quad (1.10)$$

Finally we give in this section a theorem concerning multiplication operations for n-fractions. Although the proof is relatively simple (it can be given using a generalization of the Euler–Minding formulas; see de Bruin [2]) it is omitted here.

THEOREM 1.3. *Consider an n-fraction of the form* (1.2) *with* νth *approximant numerators* $A_\nu^{(i)}$ ($i = 1,\dots, n$) *and denominators* B_ν ($\nu \in \mathbb{N}_o$) *and the n-fraction given by*

$$\begin{pmatrix} & a_1^{(1)}\rho_1\rho_0 \cdots \rho_{-n+1} & \cdots & a_\nu^{(1)}\rho_\nu\rho_{\nu-1} \cdots \rho_{\nu-n} & \cdots \\ b_0^{(1)}\rho_0\rho_{-1} \cdots \rho_{-n+1} & a_1^{(2)}\rho_1\rho_0 \cdots \rho_{-n+2} & \cdots & a_\nu^{(2)}\rho_\nu\rho_{\nu-1} \cdots \rho_{\nu-n+1} & \cdots \\ b_0^{(2)}\rho_0\rho_{-1} \cdots \rho_{-n+2} & \vdots & & \vdots & \\ \vdots & & & & \\ b_0^{(n-1)}\rho_0\rho_{-1} & a_1^{(n)}\rho_1\rho_0 & \cdots & a_\nu^{(n)}\rho_\nu\rho_{\nu-1} & \cdots \\ b_0^{(n)}\rho_0 & b_1\rho_1 & \cdots & b_\nu\rho_\nu & \cdots \end{pmatrix} \quad (1.11)$$

with νth *numerators* $\tilde{A}_\nu^{(i)}$ ($i = 1,\dots, n$) *and denominators* $\tilde{B}_\nu$ ($\nu \in \mathbb{N}_o$), *where the complex numbers* $\rho_{-n+1}, \rho_{-n+2},\dots, \rho_0, \rho_1,\dots,$ *all are different from zero. Then* (i)

$$\left.\begin{aligned} \tilde{A}_\nu^{(i)} &= \rho_\nu\rho_{\nu-1} \cdots \rho_1\rho_0 \cdots \rho_{-n+i} A_\nu^{(i)} \quad (i = 1,\dots, n) \\ \tilde{B}_\nu &= \rho_\nu\rho_{\nu-1} \cdots \rho_1 B_\nu (\tilde{B}_0 = B_0 = 1) \end{aligned}\right\}, \qquad \nu \in \mathbb{N}_0 .$$

(ii) *Under the condition* $a_\nu^{(1)} \neq 0$ $(\nu \in \mathbb{N})$, *an n-fraction with approximant numerators*

$\tilde{A}_\nu^{(i)}$ $(i = 1,\dots, n)$ *and denominators* $\tilde{B}_\nu$ $(\nu \in \mathbb{N}_o)$ *satisfying*

$\tilde{A}_\nu^{(i)}/\tilde{B}_\nu = \rho_0\rho_{-1} \cdots \rho_{-n+i}A_\nu^{(i)}/B_\nu$ $(i = 1,\dots, n;\ \nu \in \mathbb{N}_o)$ *for certain*

$\rho_0, \rho_{-1},\dots, \rho_{-n+1}$ *all different from zero, has the form* (1.11).

Remark 1.2. When $\rho_{-n+1} = \rho_{-n+2} = \cdots = \rho_{-1} = \rho_0 = 1$, the n-fractions (1.2) and (1.11) have the same sequence of approximant n-tuples; they are called *equivalent*; this gives rise to an equivalence relation. From this we can see that in these circumstances the multiplication operation does not influence the convergence behavior (see Section 2 for a rigorous definition) and therefore can be used to lead to an n-fraction with coefficients that are much more tractable.

2. The Concept of Convergence

Consider terminating and nonterminating n-fractions

$$\begin{pmatrix} & a_1^{(1)} & \cdots & a_\nu^{(1)} \\ b_0^{(1)} & \vdots & & \vdots \\ \vdots & & & \\ \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} \\ b_0^{(n)} & b_1 & \cdots & b_\nu \end{pmatrix} \text{(A)}; \quad \begin{pmatrix} & a_1^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ b_0^{(1)} & \vdots & & \vdots & \\ \vdots & & & & \\ \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_0^{(n)} & b_1 & \cdots & b_\nu & \cdots \end{pmatrix} \text{(B)} \qquad (2.1)$$

and denote the approximant n-tuples (from Theorem 1.1) by $\{A_k^{(i)}/B_k\}_{i=1}^n$ for $k = 0, 1,\dots, \nu$ and $k \in \mathbb{N}_o$, respectively.

Definition 2.1. A. The n-fraction (2.1A) is called *undefined* when $B_\nu = 0$; otherwise it is called *defined* and the values are given by the notation (1.3).

B. The n-fraction (2.1B) is called *convergent* when the limits $\lim_{\nu\to\infty} A_\nu^{(i)}/B_\nu$ exist for $i = 1,\dots, n$; otherwise it is called *divergent*. A convergent n-fraction with limit n-tuple $\xi_0^{(1)},\dots, \xi_0^{(n)}$ is given by the notation (1.3) with the nonterminating array on the right-hand side.

Remark 2.1. If the n-fraction (2.1B) is convergent, we have $B_\nu \neq 0$ for $\nu \geqslant \nu_0$. The following theorems are needed to give the proofs of the results of Section 5.

Theorem 2.1. *When* $a_1^{(1)}, a_2^{(1)},\dots, a_\lambda^{(1)} \neq 0$, *any two of the following formulas imply the third one.*

$$\begin{pmatrix} \xi_0^{(1)} \\ \vdots \\ \xi_0^{(n)} \end{pmatrix} \overset{k}{=} \begin{pmatrix} & a_1^{(1)} & \cdots & a_{\lambda-1}^{(1)} & a_\lambda^{(1)} \\ b_0^{(1)} & \vdots & & \vdots & \xi_\lambda^{(1)} \\ \vdots & & & & \vdots \\ & a_1^{(n)} & \cdots & a_{\lambda-1}^{(n)} & \vdots \\ b_0^{(n)} & b_1 & \cdots & b_{\lambda-1} & \xi_\lambda^{(n)} \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} \xi_\lambda^{(1)} \\ \vdots \\ \xi_\lambda^{(n)} \end{pmatrix} \overset{k}{=} \begin{pmatrix} & a_{\lambda+1}^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ a_\lambda^{(2)} & \vdots & & \vdots & \\ \vdots & & & & \\ a_\lambda^{(n)} & a_{\lambda+1}^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_\lambda & b_{\lambda+1} & \cdots & b_\nu & \cdots \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} \xi_0^{(1)} \\ \vdots \\ \xi_0^{(n)} \end{pmatrix} \overset{k}{=} \begin{pmatrix} & a_1^{(1)} & \cdots & a_\lambda^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ b_0^{(1)} & \vdots & & \vdots & & \vdots & \\ \vdots & & & & & & \\ & a_1^{(n)} & \cdots & a_\lambda^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_0^{(n)} & b_1 & \cdots & b_\lambda & \cdots & b_\nu & \cdots \end{pmatrix}. \tag{2.4}$$

The n-fractions (2.3) *and* (2.4) *must be both non-terminating or must be both terminating, in which case the index of the last column must be the same.*

Proof. For an elementary proof analogous to that of the theorem for $n = 1$ in Perron [6] see de Bruin [2] (compare Perron [5]). ∎

THEOREM 2.2. *Let an n-fraction K of the form* (1.2) *and an n-fraction L of the form* (1.11), *with* $\rho_\nu \neq 0$ *for all* ν, *be given; either both terminating* (*with same index in the last column*) *or both nonterminating. Then*

K defined/convergent $\Leftrightarrow$ *L defined/convergent.*

Proof. This is an immediate consequence of Theorem 1.3.

Remark 2.2. In the case that the coefficients appearing in an n-fraction depend on for instance a variable x, the notion of uniform convergence on a set G of x-values is defined in the usual way: an n-fraction (1.2) is *uniformly convergent* in x on G to an n-tuple of functions $\xi_0^{(1)}(x),\dots, \xi_0^{(n)}(x)$ if there exists for each $\epsilon > 0$ an N, such that $| \xi_0^{(i)}(x) - A_\nu^{(i)}(x)/B_\nu(x)| < \epsilon$ $(i = 1,\dots, n)$ for $\nu > N$ and all $x \in G$.

For an ordinary continued fraction converging to (if it is nonterminating) or having the value (if it is terminating) ξ_0, we have a very simple inversion theorem (see Perron [6]).

If

$$\xi_0 = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots,$$

then

$$\text{(a)} \quad \xi_0 \neq 0 \Rightarrow \frac{1}{\xi_0} = \frac{1|}{|b_0} + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots,$$

$$\text{(b)} \quad \xi_0 = 0 \Rightarrow \frac{1|}{|b_0} + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$$

is inessentially divergent (nonterminating case)/undefined (terminating case). The same can be done for n-fractions; in this case we have a more complex situation due to the "intertwining" fractions of (1.1).

THEOREM 2.3. *Let an n-fraction with values* $\xi_0^{(1)},..., \xi_0^{(n)}$ *as in* (2.4) *be given, either terminating or nonterminating. Then we have for each* $j \in \{1,..., n\}$, *fixed,*

$$\xi_0^{(j)} \neq 0 \Rightarrow \begin{pmatrix} \xi_0^{(j+1)}/\xi_0^{(j)} \\ \vdots \\ \xi_0^{(n)}\ /\xi_0^{(j)} \\ 1\ /\xi_0^{(j)} \\ \vdots \\ \xi_0^{(j-1)}/\xi_0^{(j)} \end{pmatrix} \stackrel{k}{=} \begin{pmatrix} & 1 & \cdots & 1 & 1 & a_1^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ 0 & 0 & \cdots & 0 & b_0^{(1)} & \vdots & & \vdots & \\ \vdots & \vdots & & \vdots & \vdots & & & & \\ \vdots & \vdots & & \vdots & \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ 0 & \underbrace{0 \ \cdots \ 0}_{n-j \text{ columns}} & & & b_0^{(n)} & b_1 & \cdots & b_\nu & \cdots \end{pmatrix}. \tag{2.5}$$

Proof. See de Bruin [2]. ∎

DEFINITION 2.2. Let $j \in \{1,..., n\}$ be fixed.

(a) A terminating n-fraction (with column index ν) is called *j-undefined* if $B_\nu = 0$, $A_\nu^{(j)} \neq 0$.

(b) A nonterminating n-fraction is called *j-inessentially divergent* if $\lim_{\nu\to\infty} B_\nu/A_\nu^{(j)} = 0$, $\lim_{\nu\to\infty} A_\nu^{(i)}/A_\nu^{(j)}$ exists for $i = 1,..., n$, $i \neq j$. Using these concepts, we can give a theorem like Theorem 2.1.

THEOREM 2.4. *Let* $j \in \{1,..., n\}$ *be fixed. When* $a_1^{(1)}, a_2^{(1)},..., a_\lambda^{(1)} \neq 0$, *any two of the following formulas imply the third one.*

$$\begin{pmatrix} & a_1^{(1)} & \cdots & a_{\lambda-1}^{(1)} & a_\lambda^{(1)} \\ b_0^{(1)} & \vdots & & \vdots & \xi_\lambda^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & a_1^{(n)} & \cdots & a_{\lambda-1}^{(n)} & \vdots \\ b_0^{(n)} & b_1 & \cdots & b_{\lambda-1} & \xi_\lambda^{(n)} \end{pmatrix} \tag{2.6}$$

is j-undefined,

$$\begin{pmatrix} \xi_\lambda^{(1)} \\ \vdots \\ \xi_\lambda^{(n)} \end{pmatrix} \stackrel{k}{=} \begin{pmatrix} & a_{\lambda+1}^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ a_\lambda^{(2)} & \vdots & & \vdots & \\ \vdots & & & & \\ a_\lambda^{(n)} & a_{\lambda+1}^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_\lambda & b_{\lambda+1} & \cdots & b_\nu & \cdots \end{pmatrix} \tag{2.7}$$

is a nonterminating convergent n-fraction, and

$$\begin{pmatrix} & a_1^{(1)} & \cdots & a_\lambda^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ b_0^{(1)} & \vdots & & \vdots & & \vdots & \\ \vdots & & & & & & \\ \vdots & a_1^{(n)} & \cdots & a_\lambda^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ b_0^{(n)} & b_1 & \cdots & b_\lambda & \cdots & b_\nu & \cdots \end{pmatrix} \tag{2.8}$$

is j-inessentially divergent.

Proof. Reformulate (2.6) and (2.8) by putting

$$\begin{pmatrix} & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$\underbrace{}_{n-j\text{ columns}}$$

in front of the right-hand sides; their "final" values then become $A^{(j+1)}/A^{(j)},\ldots, A^{(n)}/A^{(j)}$, $B/A^{(j)}$, $A^{(1)}/A^{(j)},\ldots, A^{(j-1)}/A^{(j)}$ (when $A^{(1)}/B,\ldots, A^{(n)}/B$ are the final approximants of (2.6)) and the limits of the sequences $\{A_\nu^{(j+1)}/A_\nu^{(j)}\}$, $\ldots$, $\{A_\nu^{(n)}/A_\nu^{(j)}\}$, $\{B_\nu/A_\nu^{(j)}\}$, $\{A_\nu^{(1)}/A_\nu^{(j)}\},\ldots, \{A_\nu^{(j-1)}/A_\nu^{(j)}\}$ (if they exist). Then apply Theorem 2.1. ∎

3. C-n-Fractions and Formal Power Series

In this section we consider n-fractions (1.2) in which the a's and b's depend upon a complex variable x.

Definition 3.1. A *C-n-fraction* is an n-fraction of form (1.2) with $b_0^{(i)} = b_{i,0}x^{r(i,0)}$ $(i = 1,\ldots, n-1)$, $b_0^{(n)} = 1$, $b_\nu = 1$ $(\nu \in \mathbb{N})$, $a_\nu^{(i)} = a_{i,\nu}x^{r(i,\nu)}$ $(i = 1,\ldots, n;\ \nu \in \mathbb{N})$, where the coefficients $b_{1,0},\ldots, b_{n-1,0}$, $a_{i,\nu}$ $(i = 1,\ldots, n;\ \nu \in \mathbb{N})$ are complex numbers with $a_{1,\nu} \neq 0$ $(\nu \in \mathbb{N})$ and the exponents $r(1, 0),\ldots, r(n-1, 0)$, $r(i, \nu)$ $(i = 1,\ldots, n;\ \nu \in \mathbb{N})$ are nonnegative integers satisfying the conditions

$$\begin{aligned} &0 \leqslant r(j, 0) < r(j, 1) < r(j-1, 2) < \cdots \\ &\qquad < r(2, j-1) < r(1, j) \qquad (j = 1,..., n-1) \\ &1 \leqslant r(n, \nu) < r(n-1, \nu+1) < r(n-2, \nu+2) < \cdots \\ &\qquad < r(2, \nu+n-2) < r(1, \nu+n-1) \qquad (\nu \in \mathbb{N}) \end{aligned} \tag{3.1}$$

The C-n-fraction is called *without zero entries* if *all* coefficients $b_{i,0}$ $(i = 1,..., n-1)$, $a_{i,\nu}$ $(i = 1,..., n;\ \nu \in \mathbb{N})$ differ from zero.

Remark 3.1. Conditions (3.1) imply $r(i, \nu) \geqslant \min(n+1-i, \nu)$ $(i = 1,..., n;\ \nu \in \mathbb{N})$.

Remark 3.2. The numerators and denominators of the approximants of a C-n-fraction are polynomials in x with $A_\nu^{(n)}(0) = B_\nu(0) = 1$ $(\nu \in \mathbb{N}_o)$.

C-n-fractions can be seen as a direct generalization of an ordinary C-fraction (corresponding to a formal power series; see Perron [7]). Because a C-n-fraction gives rise to n sequences of rational functions, it is obvious that, if a correspondence with formal power series can be found, an n-tuple of formal power series has to be considered. In the following, some theorems concerning this correspondence are given without proofs; for these we refer to de Bruin [2].

THEOREM 3.1. *A terminating C-n-fraction represents an n-tuple of rational functions of x; the C-n-fraction is undefined in the poles of these rational functions.*

THEOREM 3.2. *To each C-n-fraction without zero entries there corresponds an n-tuple of formal power series* $f^{(i)}(x) = c_0^{(i)} + c_1^{(i)}x + c_2^{(i)}x^2 + \cdots$ *with* $c_0^{(n)} = 1$, *in the following way*:

(i) *The C-n-fraction is nonterminating.*

The MacLaurin series for $A_\nu^{(i)}(x)/B_\nu(x)$ *agrees with* $f^{(i)}(x)$ $(i = 1,..., n)$ *term by term, to a power of x that increases with* ν, *but at least up to and including* x^ν $(\nu \in \mathbb{N}_o)$.

(ii) *The C-n-fraction terminates with column with index* λ.

The MacLaurin series for $A_\nu^{(i)}(x)/B_\nu(x)$ *agrees with that of* $A_\lambda^{(i)}(x)/B_\lambda(x)$ $(i = 1,..., n)$ *term by term, to a power of x that increases with* ν, *but at least up to and including* x^ν $(\nu = 0, 1,..., \lambda)$.

THEOREM 3.3. *To each n-tuple of formal power series* $f^{(1)}(x),..., f^{(n)}(x)$ *with* $f^{(n)}(0) = 1$, *for which the functions* $1, f^{(1)},..., f^{(n)}$ *are linearly independent over* $\mathbb{C}[x]$, *there corresponds a nonterminating C-n-fraction without zero* *entries. The C-n-fraction in question can be found by the following formal construction*

$$f^{(1)}(x) = b_{1,0}x^{r(1,0)} + \frac{a_{1,1}x^{r(1,1)}}{f_1^{(n)}(x)}$$

$$(b_{1,0}\,,\, a_{1,1} \neq 0;\; 0 \leqslant r(1,0) < r(1,1) \text{ minimal};\, f_1^{(n)}(0) = 1),$$

$$f^{(i)}(x) = b_{i,0}x^{r(i,0)} + \frac{f_1^{(i-1)}(x)}{f_1^{(n)}(x)}$$

$$(b_{i,0} \neq 0;\; r(i,0) \text{ minimal}) \text{ for } i = 2,\dots, n-1,$$

$$f^{(n)}(x) = 1 + \frac{f_1^{(n-1)}(x)}{f_1^{(n)}(x)}\,.$$

Once $f_\nu^{(1)},\dots, f_\nu^{(n)}$ with $f_\nu^{(n)}(0) = 1$, have been found, the formal power series $f_{\nu+1}^{(1)},\dots, f_{\nu+1}^{(n)}$ follow from

$$f_\nu^{(1)}(x) = a_{2,\nu}x^{r(2,\nu)} + \frac{a_{1,\nu+1}x^{r(1,\nu+1)}}{f_{\nu+1}^{(n)}(x)}$$

$$(a_{2,\nu}\,,\, a_{1,\nu+1} \neq 0;\; r(2,\nu) < r(1,\nu+1) \text{ minimal};\, f_{\nu+1}^{(n)}(0) = 1),$$

$$f_\nu^{(i)}(x) = a_{i+1,\nu}x^{r(i+1,\nu)} + \frac{f_{\nu+1}^{(i-1)}(x)}{f_{\nu+1}^{(n)}(x)}$$

$$(a_{i+1,\nu} \neq 0;\; r(i+1,\nu) \text{ minimal}) \text{ for } i = 2,\dots, n-1,$$

$$f_\nu^{(n)}(x) = 1 + \frac{f_{\nu+1}^{(n-1)}(x)}{f_{\nu+1}^{(n)}(x)}\,.$$

EXAMPLE 3.1. Consider the functions e^x, $-\ln(1-x)/x$. Because they satisfy the conditions of Theorem 3.3, the construction leads to a nonterminating C-2-fraction. The first columns look like

$$\begin{pmatrix} & x & x^2/12 & 5x^2/6 & -37x^2/300 & \cdots \\ 1 & x/2 & -x/2 & -3x/2 & 19x/25 & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \end{pmatrix}.$$

THEOREM 3.4. *Let $\mathfrak{F}$ be the set of all n-tuples of formal power series in an indeterminate x with complex coefficients and let $\mathfrak{F}_o$ be the subset of the n-tuples for which $\{1, f^{(1)},\dots, f^{(n)}\}$ is linearly independent over $\mathbb{C}[x]$ and $f^{(n)}(0) = 1$; $\mathfrak{C}$ is the set of nonterminating C-n-fractions without zero entries. From Theorem 3.2 we derive a mapping $\psi : \mathfrak{C} \to \mathfrak{F}$ and from Theorem 3.3 a mapping $\phi : \mathfrak{F}_o \to \mathfrak{C}$ (see Diagram 3.1). Then: $\psi\phi\,|_{\mathfrak{F}_0} = id_{\mathfrak{F}_0}$.*

$$
\begin{tikzcd}
\mathfrak{F}_0 \arrow[d, "\phi"'] \arrow[r, phantom, "\subset"] & \mathfrak{F} \\
\phi\mathfrak{F}_0 \arrow[r, phantom, "\subset"] & \mathfrak{C} \arrow[u, "\psi"']
\end{tikzcd}
$$

Diagram 3.1

Remark 3.3. The matter of what happens when the construction ϕ is applied to an n-tuple which is linearly dependent over $\mathbb{C}[x]$ is not discussed in detail. It is possible to adapt the construction ϕ from Theorem 3.3 in such a way that there corresponds a "C-n-fraction" (of which coefficients can be zero!) to each n-tuple taken from $\mathfrak{F}$. The changes are given below.

A. $f_\nu^{(i)}(x)$ is a monomial for a certain $i \in \{2,..., n\}$, $\nu \in \mathbb{N}$. We get $f_{\nu+1}^{(i-1)} = f_{\nu+2}^{(i-2)} = \cdots = f_{\nu+i-1}^{(1)} = 0$, thus $a_{i-j,\nu+1+j} = 0$ ($j = 0, 1,..., i-1$). In the n-fraction

$$
\left(\begin{array}{cccc:c}
 & & & 0 & \\
 & & 0 & & \\
 & 0 & & & \\
 \cdot^{\cdot^{\cdot}} & & & & \\
\end{array}\right).
$$

B. $f_\mu^{(1)}(x)$ is a monomial for a certain $\mu \in \mathbb{N}$ (or $= 0$, from A). Take $a_{1,\nu} = 0$ for $\nu \geqslant \mu + 1$ and change the construction into

$$
f_\nu^{(2)}(x) = a_{3,\nu} x^{r(3,\nu)} + \frac{a_{2,\nu+1} x^{r(2,\nu+1)}}{f_{\nu+1}^{(n)}(x)}
$$

$$
(a_{3,\nu}, a_{2,\nu+1} \neq 0;\ r(3, \nu) < r(2, \nu+1) \text{ minimal};\ f_{\nu+1}^{(n)}(0) = 1),
$$

$$
f_\nu^{(i)}(x) = a_{i+1,\nu} x^{r(i+1,\nu)} + \frac{f_{\nu+1}^{(i-1)}(x)}{f_{\nu+1}^{(n)}(x)}
$$

$$
(a_{i+1,\nu} \neq 0;\ r(i+1, \nu) \text{ minimal}) \text{ for } i = 3,..., n-1,
$$

$$
f_\nu^{(n)}(x) = 1 + \frac{f_{\nu+1}^{(n-1)}(x)}{f_{\nu+1}^{(n)}(x)}
$$

for $\nu \geqslant \mu$ (i.e., a construction as if we have an $(n-1)$-tuple of formal power series!). In the n-fraction

$$
\left(\begin{array}{c:cccc}
 & 0 & 0 & 0 & \cdots \\
 & & & & \\
 & & & & \\
\end{array}\right).
$$

After n of these changes the n-fraction terminates and furthermore there must exist n linearly independent dependency relations for $1, f^{(1)},\ldots, f^{(n)}$. This shows that $f^{(1)},\ldots, f^{(n)}$ are the MacLaurin series for an n-tuple of rational functions. The converse is also true!

THEOREM 3.5. *For the MacLaurin series for an n-tuple of rational functions, the construction ϕ of Theorem* 3.3 *adapted according to Remark* 3.3, *leads to a terminating n-fraction.*

EXAMPLE 3.2. Consider the functions $1/(1-x)$, $1/(1-x^2)$ and $1/(1-x^3)$ and apply the adapted construction ϕ.

$$\frac{1}{1-x} = 1 + \frac{x}{1-x}, \qquad \frac{1}{1-x^2} = 1 + \frac{x/(1+x)}{1-x},$$

$$\frac{1}{1-x^3} = 1 + \frac{x^3/(1+x+x^2)}{1-x}; \qquad \frac{x^2}{1+x} = x^2 + \frac{-x^3}{1+x},$$

$$\frac{x^3}{1+x+x^2} = x^3 + \frac{-x^4(1+x)^2/(1+x+x^2)}{1-x},$$

$$1-x = 1 + \frac{-x(1+x)}{1+x}; \qquad \frac{-x^4(1+x)^2}{1+x+x^2} = -x^4 + \frac{-x^5}{1+x+x^2},$$

$$-x-x^2 = -x + \frac{-x^2(1+x+x^2)}{1+x+x^2}, \qquad 1+x = 1 + \frac{x(1+x+x^2)}{1+x+x^2};$$

$$-x^2(1+x+x^2) = -x^2 + \frac{-x^3}{1/(1+x)},$$

$$x(1+x+x^2) = x + \frac{x^2}{1/(1+x)}, \qquad 1+x+x^2 = 1 + \frac{x}{1/(1+x)}.$$

$x^2 = x^2 + \dfrac{0}{\cdots}$ a monomial; $f_5^{(3)}$ follows from the next line,

$x = x + \dfrac{0}{\cdots}$ again a monomial; $f_5^{(3)}$ follows from the next line,

$1/(1+x) = 1 + (-x)/(1+x)$.

There is now only one function left to go on with!

$$1 + x = 1 + x/1,$$

$1 = 1 + 0/1,$ the construction terminates.

We have the 3-fraction

$$\begin{pmatrix} & x & -x^3 & -x^5 & -x^3 & 0 & 0 \\ 1 & x^2 & -x^4 & -x^2 & x^2 & 0 & 0 \\ 1 & x^3 & -x & x & x & -x & x \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Calculation of the approximant triples gives

$$\begin{aligned}
&A^{(1)}/B : 1,\ 1 + x,\ 1/(1-x),\ 1/(1-x),\ 1/(1-x),\\
&A^{(2)}/B : 1,\ 1 + x^2,\ 1 + x^2,\ 1/(1-x^2),\ 1/(1-x^2),\\
&A^{(3)}/B : 1,\ 1 + x^3,\ 1 + x^3,\ 1 + x^3,\ (1 + x^3 + x^6)/\{(1+x)(1-x^2)\},
\end{aligned}$$

$$\begin{aligned}
&1/(1-x), && 1/(1-x),\\
&1/(1-x^2), && 1/(1-x^2),\\
&(1 + x^3 + x^6)/(1-x^2), && 1/(1-x^3).
\end{aligned}$$

4. Regular C-n-Fractions

As in the case of the ordinary C-fractions, there is a special class which is closely connected with sequences of Padé approximants on a stepline in the Padé table for an n-tuple of functions (see [2, 3]). This special class gives rise to convergence theorems which reduce to classical ones for $n = 1$.

Definition 4.1. A C-n-fraction is called *regular* if it is nonterminating and moreover satisfies

$$\begin{aligned}
b_{i,0} &= 1 \qquad (i = 1,\ldots, n-1),\\
a_{i,\nu} &\neq 0 \qquad (i = 1,\ldots, n;\ \nu \in \mathbb{N}),\\
r(i, 0) &= 0 \qquad (i = 1,\ldots, n-1),\\
r(i, \nu) &= \min(n + 1 - i, \nu) \qquad (i = 1,\ldots, n;\ \nu \in \mathbb{N}).
\end{aligned} \tag{4.1}$$

Remark 4.1. In a certain way, regularity is connected with the minimality of the construction from Theorem 3.3 and gaps in the power series $f^{(1)},\ldots, f^{(n)}$.

Remark 4.2. Schematically the degrees of the entries in a regular C-n-fraction can be given by

$$\begin{pmatrix} & 1 & 2 & 3 & \cdots & n-1 & n & \cdots & n & \cdots \\ 0 & 1 & 2 & 3 & \cdots & n-1 & n-1 & \cdots & n-1 & \cdots \\ \vdots & \vdots & & & & & & & & \\ 0 & 1 & 2 & 3 & \cdots & 3 & 3 & \cdots & 3 & \cdots \\ 0 & 1 & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 & \cdots \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots \end{pmatrix}.$$

Some properties of regular C-n-fractions are given below without proofs; for these we refer to [2].

THEOREM 4.1. *For a regular C-n-fraction we have for* $\nu \in \mathbb{N}_0$

$$\text{(a)}\quad \deg B_{(n+1)\nu+j} = \begin{cases} = n\nu & j = 0^{(*)} \\ \leqslant n\nu + j - 1 & j = 1,\dots, n \end{cases}$$

$$\text{(b)}\quad \deg A^{(i)}_{(n+1)\nu+j} = \begin{cases} = n\nu + i & j = i^{(*)} \\ \leqslant n\nu + j & j = 1,\dots, i-1 \quad (i = 1,\dots, n) \\ \leqslant n\nu + j - 1 & j = i+1,\dots, n+1 \end{cases}$$

In the cases$^{(*)}$, the highest power of x has the coefficient

(a) $a_{1,n+1}a_{1,2n+2} \cdots a_{1,\nu(n+1)}$.

(b) $a_{1,i}a_{1,n+1+i} \cdots a_{1,\nu(n+1)+i} \qquad (i = 1,\dots, n)$.

THEOREM 4.2. *Let* $f^{(1)},\dots, f^{(n)}$ *be the n-tuple of formal power series, constructed by* ψ *from Theorem* 3.4 *from a regular C-n-fraction with approximants* $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{i=1}^{n}$ $(\nu \in \mathbb{N}_0)$. *Then there exists for each* $\nu \in \mathbb{N}$ *a* $k \in \{1,\dots, n\}$ *with*

$$f^{(k)}(x) - A_\nu^{(k)}(x)/B_\nu(x) = dx^{\nu+1} + \text{higher powers}, \qquad d \neq 0.$$

THEOREM 4.6. *For a regular C-n-fraction K, we have* $\psi K \in \mathfrak{F}_0$.

THEOREM 4.7. *Let* $\mathfrak{C}_r \subset \mathfrak{C}$ *be the set of all regular C-n-fractions. Then*

$$\phi\psi\,|_{\mathfrak{C}_r} = id_{\mathfrak{C}_r}.$$

Remark 4.3. Together with the result of Theorem 3.4 we have the situation of Diagram 4.1. With $\psi^* = \psi\,|_{\mathfrak{C}_r}$, $\phi^* = \phi\,|_{\psi\mathfrak{C}_r}$

$$\psi^*\phi^* = \phi^*\psi^* = id_{\mathfrak{C}_r}.$$

$$\begin{array}{ccccc} \psi\mathfrak{C}_r & \subset & \mathfrak{F}_0 & \subset & \mathfrak{F} \\ {\scriptstyle\psi}\uparrow & & \downarrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi} \\ \mathfrak{C}_r & \subset & \phi\mathfrak{F}_0 & \subset & \mathfrak{C} \end{array}$$

Diagram 4.1

5. Main Results

In this section we only consider C-n-fractions as in Definition 3.1 that satisfy

$$r(i, 0) = 0\ (i = 1,..., n - 1);\ \ r(i, \nu) = \min(n + 1 - i, \nu)\ \ (i = 1,..., n;\ \nu \in \mathbb{N}) \tag{5.1}$$

(i.e., C-n-fractions in which the exponents behave like those in a regular C-n-fraction; the coefficients, apart from the 1st and $(n + 1)$st row may be zero)

Let ρ be the *unique* real number defined by

$$\rho^n + 2^{-1}\rho^{n-1} + 2^{-2}\rho^{n-2} + \cdots + 2^{-n+1}\rho - 2^{-n-1} = 0, \qquad \tfrac{1}{6} < \rho \leqslant \tfrac{1}{4} \tag{5.2}$$

($\rho = \rho(n)$ satisfies: $\rho(1) = \frac{1}{4}$, $\rho(n + 1) < \rho(n)$ for all n, $\rho(n) \to \frac{1}{6}$ for $n \to \infty$)

Theorem 5.1. *Let* $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{i=1}^n$ $(\nu \in \mathbb{N})$ *be the approximant n-tuples of a C-n-fraction that satisfies* (5.1) *and let the following hold*

$$a_i = \sup_{\nu \geqslant 2} |a_{i,\nu}| < \infty \qquad (i = 1,..., n) \tag{5.3}$$

Then the n sequences $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{\nu=0}^\infty$ $(i = 1,..., n)$ *converge to an n-tuple of analytic functions, uniformly in x on each compact subset of the domain*

$$\mathfrak{D} = \{x \in \mathbb{C} \mid |x| < \rho \cdot \min_i(a_i^{-1/(n+1-i)})\} \tag{5.4}$$

(*if* $a_i = 0$ *for a certain i, it has to be omitted*).

Theorem 5.2. *Consider a C-n-fraction as in Theorem* 5.1 *with*

$$a_i = \limsup_{\nu \to \infty} |a_{i,\nu}| < \infty \qquad (i = 1,..., n) \tag{5.5}$$

Then the n sequences $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{\nu=0}^\infty$ $(i = 1,..., n)$ *converge to an n-tuple of functions that are analytic at* $x = 0$ *and meromorphic on the domain* $\mathfrak{D}$ *from* (5.4). *In the poles of the limit functions the C-n-fraction is j-inessentially divergent for at least one j,* $j \in \{1,..., n\}$.

Theorem 5.3. *Let* $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{i=1}^n$ $(\nu \in \mathbb{N}_0)$ *be the approximants of a* regular *C-n-fraction K of which the coefficients satisfy* (5.3). *Then* $\lim_{\nu\to\infty} A_\nu^{(i)}(x)/B_\nu(x) = g^{(i)}(x)$ $(i = 1,..., n)$, *uniformly in x on each compact subset of the domain* $\mathfrak{D}$ *from* (5.4). *The functions* $g^{(1)},..., g^{(n)}$ *are analytic on* $\mathfrak{D}$: $\{1, g^{(1)},..., g^{(n)}\}$ *is linearly independent over* $\mathbb{C}[x]$.

Furthermore, if $\psi K = (f^{(1)},..., f^{(n)})$, *then* $f^{(i)} \equiv g^{(i)}$ $(i = 1,..., n)$ *on* $\mathfrak{D}$.

THEOREM 5.4. *Consider a situation as in Theorem 5.2, arising from a* regular *C-n-fraction K. Then* $\lim_{\nu\to\infty} A_\nu^{(i)}(x)/B_\nu(x) = g^{(i)}(x)$ $(i = 1,..., n)$, *where the functions* $g^{(1)},..., g^{(n)}$ *are analytic at* $x = 0$ *and meromorphic on* $\mathfrak{D}$ *from* (5.4); $\{1, g^{(1)},..., g^{(n)}\}$ *is linearly independent over* $\mathbb{C}[x]$. *In the poles of* $g^{(1)},..., g^{(n)}$ *the C-n-fraction is j-inessentially divergent for at least one j,* $j \in \{1,..., n\}$. *Furthermore, if* $\psi K = (f^{(1)},..., f^{(n)})$, *and* $\mathfrak{E}$ *is the domain of meromorphy of* $f^{(1)},..., f^{(n)}$, *then we have* $\mathfrak{D} \subset \mathfrak{E}$ *and* $f^{(i)} \equiv g^{(i)}$ $(i = 1,..., n)$ *on* $\mathfrak{D}$.

Remark 5.1. For $n = 1$ Theorems 5.3 and 5.4 reduce to the theorem due to van Vleck [14] and Pringsheim [8] (see also Perron [7, p. 148]), given in the Introduction.

EXAMPLE 5.1. Consider the functions $(1 - x)^{1/2}$, $(1 - x)^{1/4}$. They have a regular C-2-fraction with

$$a_{1,1} = 1,\ a_{1,3\nu+1} = (\nu + \tfrac{1}{2})(\nu + \tfrac{1}{4})/\{(3\nu - 1)\, 3\nu \cdot (3\nu + 1)\} \qquad (\nu \in \mathbb{N}),$$
$$a_{1,2} = \tfrac{1}{8},\ a_{1,3\nu+2} = (\nu + \tfrac{1}{2})(\nu + \tfrac{1}{4})/\{3\nu(3\nu + 1)(3\nu + 2)\} \qquad (\nu \in \mathbb{N}),$$
$$a_{1,3\nu+3} = (\nu + 1)(\nu + \tfrac{3}{4})(\nu + \tfrac{1}{2})/\{(3\nu + 1)(3\nu + 2)(3\nu + 3)\} \qquad (\nu \in \mathbb{N}_o)$$

and

$$a_{2,1} = 1,\ a_{2,3\nu+1} = -(3\nu + \tfrac{3}{4})/\{3 \cdot (3\nu + 1)\} \qquad (\nu \in \mathbb{N}),$$
$$a_{2,2} = -\tfrac{1}{4},\ a_{2,3\nu+2} = -(3\nu^2 + \tfrac{9}{4}\nu + \tfrac{1}{2})/\{(3\nu + 1)(3\nu + 2)\} \qquad (\nu \in \mathbb{N}),$$
$$a_{2,3\nu+3} = -(\nu + 1)/(3\nu + 2) \qquad (\nu \in \mathbb{N}_o)$$

(see de Bruin [2]). Then $a_1 = 1/27$, $a_2 = \frac{1}{3}$ and thus we get convergence of the two sequences of approximants to the starting functions (uniformly in x on compact subsets) on the domain $\mathfrak{D} = \{x \in \mathbb{C} \mid |x| < 3(3^{1/2} - 1)/4\}$; there are no poles in $\mathfrak{D}$! Because $3(3^{1/2} - 1)/4 = 0{,}549...$, we see that the convergence theorem is not yet best possible.

EXAMPLE 5.2. Let $\beta_1,..., \beta_n \in \mathbb{R}^+$ (i.e., > 0) and consider the hypergeometric series given below.

$${}_0F_n(\beta_1,..., \beta_n; x) = \sum_{j=0}^{\infty} x^j \Big/ \Big\{ j! \prod_{i=1}^{n} (\beta_i)_j \Big\}$$

with $(\beta)_0 = 1$, $(\beta)_j = \beta(\beta + 1) \cdots (\beta + j - 1)$ $(j \in \mathbb{N})$. The numbers $B_k(\beta_1,..., \beta_n)$ $(k = 1,..., n)$ are defined by

$$\prod_{i=1}^{n} (x + \beta_i) - \prod_{i=1}^{n} \beta_i = \sum_{k=1}^{n} B_k(\beta_1,..., \beta_n)(x - k - 1)_k \qquad (B_n \equiv 1).$$

Now let $\alpha_1,\ldots,\alpha_n \in \mathbb{R}^+\backslash\{1,\ldots,n-1\}$ be given, then the functions

$$f^{(n-j)}(x) = \sum_{k=j}^{n} \frac{B_k(\alpha_1 - j,\ldots,\alpha_n - j)}{\prod_{i=1}^{n}(\alpha_i - j)_{k+1}} \times x^{k-j} \frac{{}_0F_n(\alpha_1 + k - j + 1,\ldots,\alpha_n + k - j + 1; x)}{{}_0F_n(\alpha_1 + 1,\ldots,\alpha_n + 1; x)} \qquad (j = 1,\ldots,n-1),$$

$$f^{(n)}(x) = \frac{{}_0F_n(\alpha_1,\ldots,\alpha_n; x)}{{}_0F_n(\alpha_1 + 1,\ldots,\alpha_n + 1; x)} \tag{5.6}$$

have a regular C-n-fraction with coefficients

$$\begin{aligned} b_{n-k,0} &= B_k(\alpha_1 - k,\ldots,\alpha_n - k)\Big/\prod_{i=1}^{n}(\alpha_i - k)_{k+1} \qquad (k = 1,\ldots,n-1), \\ a_{n+1-k,\nu} &= B_k(\alpha_1 + \nu - k,\ldots,\alpha_n + \nu - k)\Big/\prod_{i=1}^{n}(\alpha_i + \nu - k)_{k+1} \qquad (k = 1,\ldots,n;\ \nu \in \mathbb{N}). \end{aligned} \tag{5.7}$$

Because $a_j = 0$ $(j = 1,\ldots,n)$, the regular C-n-fraction converges to an n-tuple of functions analytic at $x = 0$ and meromorphic on $\mathbb{C}$; the limit functions equal the functions (5.6) on the domain where the latter are meromorphic.

Remark 5.2. Of course it is possible to derive convergence results for general C-n-fractions as can be seen from the method of proof in Section 6. The quantities that determine the domain of convergence do not have a simple form as in the theorems above; one could derive a theorem in which, instead of $a_i^{-1/(n+1-i)}$ $(i = 1,\ldots,n)$, the following quantities appear

$$\eta_i = \inf\{|a_{i,\nu}|^{-1/r(i,\nu)} \mid a_{i,\nu} \neq 0, \nu \geqslant 2\} \qquad (i = 1,\ldots,n),$$

and furthermore ρ from (5.2) has to be replaced by another number that must be chosen optimal, keeping the values $\eta_1,\ldots,\eta_n$ in mind (see Lemma 6.6 and the proof of Theorem 5.1 for the choice that leads to ρ).

6. Proof of Results

DEFINITION 6.1. $\{k_\mu\}_{\mu=-n+1}$ is the *generalized Fibonacci sequence* given by

$$\begin{aligned} &k_{-n+1} = k_{-n+2} = \cdots = k_{-1} = 0, \qquad k_0 = 1, \\ &k_\mu = k_{\mu-1} + k_{\mu-2} + \cdots + k_{\mu-n} \qquad (\mu \in \mathbb{N}). \end{aligned} \tag{6.1}$$

LEMMA 6.1. Let τ be the unique root, satisfying $\frac{1}{2} \leqslant \tau < 1$, of

$$1 - z - z^2 - \cdots - z^n = 0 \tag{6.2}$$

Then we have

$$k_\nu \sim \gamma\tau^{-\nu} \quad \text{for} \quad \nu \to \infty, \qquad \gamma = \tau^2 \prod_{j=2}^{n} (2\tau_j - 1)/(\tau_j - \tau) \neq 0, \tag{6.3}$$

where $\tau, \tau_2, \ldots, \tau_n$ are the roots of (6.2).

Proof. Either use complex function theory or the method already employed by Perron [5, Section 8]. ▮

Remark 6.1. Some properties of $\tau = \tau(n)$ are

(a) $\tau(1) = 1$, $\tau(2) = (5^{1/2} - 1)/2$; (b) $\tau(n) > \tau(n+1)$ for all n; (c) $\tau(n) \to \frac{1}{2}$ for $n \to \infty$.

LEMMA 6.2. *Consider an n-fraction of the form given below*

$$\begin{pmatrix} & a_1^{(1)} & \cdots & a_\nu^{(1)} & \cdots \\ 0 & \vdots & & \vdots & \\ \vdots & a_1^{(n)} & \cdots & a_\nu^{(n)} & \cdots \\ 0 & b_1 & \cdots & b_\nu & \cdots \end{pmatrix}; \qquad a_\nu^{(1)}, b_\nu \neq 0 \quad \text{for} \quad \nu \in \mathbb{N}. \tag{6.4}$$

Let the coefficients satisfy

$$|a_1^{(1)}/b_1| \leqslant K(p_1 - 1)/p_1, \qquad |a_1^{(i)}/b_1| \leqslant K(p_1 - 2)/p_1 \qquad (i = 2, \ldots, n),$$

$$\left|\frac{a_\nu^{(1)}}{b_{\nu-n} \cdots b_\nu}\right| \leqslant \frac{p_\nu - 1}{p_{\nu-n} \cdots p_\nu}, \qquad \left|\frac{a_\nu^{(i)}}{b_{\nu-(n+1-i)} \cdots b_\nu}\right| \leqslant \frac{p_\nu - 2}{p_{\nu-(n+1-i)} \cdots p_\nu}$$

$$(i = 2, \ldots, n) \qquad \text{for} \quad \nu \in \mathbb{N}\backslash\{1\} \tag{6.5}$$

with $p_\nu \geqslant 2$ $(\nu \in \mathbb{N})$, $K > 0$ *(b's and p's with index $\leqslant 0$ have to be omitted). Let* $\{A_\nu^{(i)}/B_\nu\}_{i=1}^n$ $(\nu \in \mathbb{N})$ *be the approximants of* (6.4) *and* $\{C_\nu^{(i)}/D_\nu\}_{i=1}^n$ $(\nu \in \mathbb{N})$ *those of the n-fraction that follows, as in* (1.10), *from* (6.4) *by multiplying with* $\rho_{-j} = 1$ $(j = 1, \ldots, n-1)$, $\rho_0 = 1/K$, $\rho_\nu = p_\nu/b_\nu$ $(\nu \in \mathbb{N})$. *Then*

(a) $A_\nu^{(i)}/B_\nu = KC_\nu^{(i)}/D_\nu \qquad (i = 1, \ldots, n;\ \nu \in \mathbb{N}).$

(b) $|D_\nu| \geqslant \sum_{\mu=0}^{\nu} k_\mu \prod_{\sigma=1}^{\nu-\mu} (p_\sigma - 1) \qquad (\nu \in \mathbb{N}_0).$

Proof. Part (a) follows from Theorem 1.3 and the choice of the ρ's.

Furthermore the multiplication operation leads to

$$|\rho_{\nu-n} \cdots \rho_\nu a_\nu^{(1)}| \leqslant p_\nu - 1,$$

$$|\rho_{\nu-(n+1-i)} \cdots \rho_\nu a_\nu^{(i)}| \leqslant p_\nu - 2 \qquad (i = 2,..., n)\ (\nu \in \mathbb{N}).$$

The D_ν then satisfy

$$\begin{aligned} |D_\nu| &= |\rho_\nu b_\nu D_{\nu-1} + \rho_\nu\rho_{\nu-1} a_\nu^{(n)} D_{\nu-2} + \cdots + \rho_{\nu-n} \cdots \rho_\nu a_\nu^{(1)} D_{\nu-n-1}| \\ &\geqslant p_\nu |D_{\nu-1}| - (p_\nu - 2)(|D_{\nu-2}| + \cdots + |D_{\nu-n}|) - (p_\nu - 1)|D_{\nu-n-1}| \end{aligned}$$

or

$$\begin{aligned} &|D_\nu| - (|D_{\nu-1}| + \cdots + |D_{\nu-n}|) \\ &\quad \geqslant (p_\nu - 1)\{|D_{\nu-1}| - (|D_{\nu-2}| + \cdots + |D_{\nu-n-1}|)\} \qquad (\nu \in \mathbb{N}). \end{aligned}$$

With $\Delta_\nu = |D_\nu| - (|D_{\nu-1}| + \cdots + |D_{\nu-n}|)$ $(\nu \in \mathbb{N})$, $\Delta_0 = |D_0| = 1$, iteration leads to

$$\Delta_\nu \geqslant \prod_{j=1}^{\nu} (p_j - 1) \qquad (\nu \in \mathbb{N}). \tag{6.6}$$

To derive a lower bound for the $|D_\nu|$, it is necessary to express them in terms of the Δ_ν; it is easy to show (induction on ν):

$$|D_\nu| = \sum_{\mu=0}^{\nu} k_\mu \Delta_{\nu-\mu} \qquad (\nu \in \mathbb{N};\ k_\mu \text{ from } (6.1)). \tag{6.7}$$

Combination of (6.6) and (6.7) leads to assertion (b). ∎

LEMMA 6.3. *Consider the following system of homogeneous linear equations in the unknowns $\lambda_1, \lambda_2, ..., \lambda_n$, depending on $\alpha, q \in \mathbb{R}^+$.*

$$1: \quad \lambda_1 = \left(1 - \frac{1}{q}\right)\frac{1}{\alpha}\lambda_1 + \left(1 - \frac{1}{q}\right)\frac{1}{\alpha}\lambda_2 + \left(1 - \frac{1}{q}\right)\frac{1}{\alpha}\lambda_3 + \cdots + \left(1 - \frac{1}{q}\right)\frac{1}{\alpha}\lambda_{n-1} + \frac{1}{\alpha}\lambda_n,$$

$$2: \quad \lambda_2 = \left(1 + \frac{1}{q}\right)\frac{1}{\alpha}\lambda_1 + \left(1 - \frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_1 + \left(1 - \frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_2 + \cdots + \left(1 - \frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_{n-2} + \frac{1}{q\alpha^2}\lambda_{n-1},$$

$$
\begin{aligned}
j\colon \quad \lambda_j &= \left(1+\frac{1}{q}\right)\frac{1}{\alpha}\lambda_{j-1} \\
&\quad + \overbrace{\left(1-\frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_{j-2} + \cdots + \left(1-\frac{1}{q}\right)\frac{1}{q^{j-2}\alpha^{j-1}}\lambda_1}^{j-2 \text{ terms}} \\
&\quad + \underbrace{\left(1-\frac{1}{q}\right)\frac{1}{q^{j-1}\alpha^j}\lambda_1 + \cdots + \left(1-\frac{1}{q}\right)\frac{1}{q^{j-1}\alpha^j}\lambda_{n-j}}_{n-j \text{ terms}} \\
&\quad + \frac{1}{q^{j-1}\alpha^j}\lambda_{n-j+1} \qquad (j = 3,\ldots, n-2),
\end{aligned}
$$

$$
\begin{aligned}
n-1\colon \quad \lambda_{n-1} &= \left(1+\frac{1}{q}\right)\frac{1}{\alpha}\lambda_{n-2} + \left(1-\frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_{n-3} + \cdots \\
&\quad + \left(1-\frac{1}{q}\right)\frac{1}{q^{n-3}\alpha^{n-2}}\lambda_1 \\
&\quad + \left(1-\frac{1}{q}\right)\frac{1}{q^{n-2}\alpha^{n-1}}\lambda_1 + \frac{1}{q^{n-2}\alpha^{n-1}}\lambda_2 ,
\end{aligned}
$$

$$
\begin{aligned}
n\colon \quad \lambda_n &= \left(1+\frac{1}{q}\right)\frac{1}{\alpha}\lambda_{n-1} + \left(1-\frac{1}{q}\right)\frac{1}{q\alpha^2}\lambda_{n-2} + \cdots \\
&\quad + \left(1-\frac{1}{q}\right)\frac{1}{q^{n-3}\alpha^{n-2}}\lambda_2 \\
&\quad + \left(1-\frac{1}{q}\right)\frac{1}{q^{n-2}\alpha^{n-1}}\lambda_1 + \frac{1}{q^{n-1}\alpha^n}\lambda_1 .
\end{aligned}
$$

In matrix notation with $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$: $A(\alpha)\,\underline{\lambda}^T = \underline{0}$. Then there exists a q_0 such that for $q > q_0$ the equation $\det A(\alpha) = 0$ has a real root α_0 with $0 < \alpha_0 < q$ and such, that the system $A(\alpha_0)\,\underline{\lambda}^T = \underline{0}$ has a solution $\underline{\lambda}$ with $\lambda_1 = 1, \lambda_i > 0$ $(i = 2,\ldots, n)$.

Proof. Because $\det A(\alpha)$ is a polynomial in α with coefficients depending on q and because

$$
\lim_{q\to\infty} \det A(\alpha) = \begin{vmatrix}
\frac{1}{\alpha}-1 & \frac{1}{\alpha} & \frac{1}{\alpha} & \cdots & \frac{1}{\alpha} \\
\frac{1}{\alpha} & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \frac{1}{\alpha} & -1
\end{vmatrix}
$$

$$
= (-1)^n \left(1 - \alpha^{-1} - \alpha^{-2} - \cdots - \alpha^{-n}\right),
$$

we see that there exists a $Q_0 > 3$, such that we have a real root $\alpha_0(q)$ of the equation $\det A(\alpha) = 0$ that is *close to* τ^{-1} (from Lemma 6.1) for $q > Q_0$; that close, to have $0 < \alpha_0 < 3 < q$. Because the system then has a nontrivial real solution for $q > Q_0$ and the solution $\underset{\sim}{\lambda}$ with $\lambda_1 = 1$, $\lambda_i = \tau^{i-1}$ $(i = 2,..., n)$ in the limiting case, this leads to the existence of a solution $\underset{\sim}{\lambda}$ with $\lambda_1 = 1$, $\lambda_i > 0$ $(i = 2,..., n)$ for $q > q_0 > Q_0$. ∎

LEMMA 6.4. *Let for* $q \geqslant 1$ *the quantities* $\delta_{\nu,j}^{(i)}$ $(i, j = 1,..., n;\ \nu \in \mathbb{N})$ *be defined by the recurrence relations* (6.8) *given below.*

$$\delta_{\nu,1}^{(i)} = (q-1)\,\delta_{\nu-1,1}^{(i)} + (q-1)\,\delta_{\nu-1,2}^{(i)} + \cdots + (q-1)\,\delta_{\nu-1,n-1}^{(i)} + q\delta_{\nu-1,n}^{(i)},$$

$$\delta_{\nu,2}^{(i)} = (q+1)\,\delta_{\nu-1,1}^{(i)} + (q-1)\,\delta_{\nu-2,1}^{(i)} + (q-1)\,\delta_{\nu-2,2}^{(i)} + \cdots + (q-1)\,\delta_{\nu-2,n-2}^{(i)} + q\delta_{\nu-2,n-1}^{(i)},$$

$$\begin{aligned}\delta_{\nu,j}^{(i)} = {} & (q+1)\,\delta_{\nu-1,j-1}^{(i)} \\ & + \underbrace{(q-1)\,\delta_{\nu-2,j-2}^{(i)} + (q-1)\,\delta_{\nu-3,j-3}^{(i)} + \cdots + (q-1)\,\delta_{\nu-j+1,1}^{(i)}}_{j-2 \text{ terms}} \\ & + \underbrace{(q-1)\,\delta_{\nu-j,1}^{(i)} + (q-1)\,\delta_{\nu-j,2}^{(i)} + \cdots + (q-1)\,\delta_{\nu-j,n-j}^{(i)}}_{n-j \text{ terms}} \\ & + q\delta_{\nu-j,n-j+1}^{(i)} \qquad (j = 3,..., n-2),\end{aligned}$$

$$\delta_{\nu,n-1}^{(i)} = (q+1)\,\delta_{\nu-1,n-2}^{(i)} + (q-1)\,\delta_{\nu-2,n-3}^{(i)} + (q-1)\,\delta_{\nu-3,n-4}^{(i)} + \cdots + (q-1)\,\delta_{\nu-n+2,1}^{(i)} + (q-1)\,\delta_{\nu-n+1,1}^{(i)} + q\delta_{\nu-n+1,2}^{(i)},$$

$$\delta_{\nu,n}^{(i)} = (q+1)\,\delta_{\nu-1,n-1}^{(i)} + (q-1)\,\delta_{\nu-2,n-2}^{(i)} + (q-1)\,\delta_{\nu-3,n-3}^{(i)} + \cdots + (q-1)\,\delta_{\nu-n+1,1}^{(i)} + q\delta_{\nu-n,1}^{(i)},$$

and initial values

$$\begin{aligned}\delta_{0,j}^{(i)} &= 1 \qquad i + j = n + 1 \\ &= 0 \qquad i + j \neq n + 1 \quad (i, j = 1,..., n), \\ \delta_{-k,j}^{(i)} &= 0 \qquad (i, j, k = 1,..., n \text{ with } j + k \leqslant n). \end{aligned} \tag{6.9}$$

Then there exists a constant N such that for $q > q_0$ (q_0 from Lemma 6.3)

$$\delta_{\nu,j}^{(i)} \leqslant N(\alpha_0 q)^\nu \qquad (i, j = 1,..., n;\ \nu \in \mathbb{N};\ \alpha_0 \textit{ from Lemma } 6.3) \tag{6.10}$$

Proof. With q_0, α_0, $\underline{\lambda} = (1, \lambda_2, \ldots, \lambda_n)$ as in Lemma 6.3 we now prove the existence of constants M_i such that

$$\delta_{\nu,j}^{(i)} \leqslant M_i \lambda_j (\alpha_0 q)^\nu \qquad (i, j = 1, \ldots, n;\ \nu \in \mathbb{N};\ q > q_0).$$

Then (6.10) follows with $N = \max_{i,j} M_i \lambda_j$. Take $M_i = (\alpha_0 q)^{-1} \max_j (\delta_{1,j}^{(i)}/\lambda_j)$ $(i = 1, \ldots, n)$, then $\delta_{1,j}^{(i)} \leqslant M_i \lambda_j (\alpha_0 q)$ for $i, j = 1, \ldots, n$. Proceed by induction on ν and consider (6.10) proven for $1, 2, \ldots, \nu - 1$; inserting the estimates in (6.8) we are led to (omit the index of α)

$$\delta_{\nu,1}^{(i)} \leqslant M_i (\alpha q)^\nu \left[\left(1 - \frac{1}{q}\right) \frac{1}{\alpha} \lambda_1 + \left(1 - \frac{1}{q}\right) \frac{1}{\alpha} \lambda_2 + \cdots \right.$$
$$\left. + \left(1 - \frac{1}{q}\right) \frac{1}{\alpha} \lambda_{n-1} + \frac{1}{\alpha} \lambda_n\right],$$

$$\delta_{\nu,2}^{(i)} \leqslant M_i (\alpha q)^\nu \left[\left(1 + \frac{1}{q}\right) \frac{1}{\alpha} \lambda_1 + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_1 + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_2 + \cdots \right.$$
$$\left. + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_{n-2} + \frac{1}{q\alpha^2} \lambda_{n-1}\right],$$

$$\delta_{\nu,j}^{(i)} \leqslant M_i (\alpha q)^\nu \left[\left(1 + \frac{1}{q}\right) \frac{1}{\alpha} \lambda_{j-1} + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_{j-2} \right.$$
$$+ \left(1 - \frac{1}{q}\right) \frac{1}{q^2\alpha^3} \lambda_{j-3} + \cdots + \left(1 - \frac{1}{q}\right) \frac{1}{q^{j-2}\alpha^{j-1}} \lambda_1$$
$$+ \left(1 - \frac{1}{q}\right) \frac{1}{q^{j-1}\alpha^j} \lambda_1 + \left(1 - \frac{1}{q}\right) \frac{1}{q^{j-1}\alpha^j} \lambda_2 + \cdots$$
$$\left. + \left(1 - \frac{1}{q}\right) \frac{1}{q^{j-1}\alpha^j} \lambda_{n-j} + \frac{1}{q^{j-1}\alpha^j} \lambda_{n-j+1}\right] \qquad (j = 3, \ldots, n-2),$$

$$\delta_{\nu,n-1}^{(i)} \leqslant M_i (\alpha q)^\nu \left[\left(1 + \frac{1}{q}\right) \frac{1}{\alpha} \lambda_{n-2} + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_{n-3} \right.$$
$$+ \left(1 - \frac{1}{q}\right) \frac{1}{q^2\alpha^3} \lambda_{n-4} + \cdots + \left(1 - \frac{1}{q}\right) \frac{1}{q^{n-3}\alpha^{n-2}} \lambda_1$$
$$\left. + \left(1 - \frac{1}{q}\right) \frac{1}{q^{n-2}\alpha^{n-1}} \lambda_1 + \frac{1}{q^{n-2}\alpha^{n-1}} \lambda_2\right],$$

$$\delta_{\nu,n}^{(i)} \leqslant M_i (\alpha q)^\nu \left[\left(1 + \frac{1}{q}\right) \frac{1}{\alpha} \lambda_{n-1} + \left(1 - \frac{1}{q}\right) \frac{1}{q\alpha^2} \lambda_{n-2} \right.$$
$$\left. + \left(1 - \frac{1}{q}\right) \frac{1}{q^2\alpha^3} \lambda_{n-3} + \cdots + \left(1 - \frac{1}{q}\right) \frac{1}{q^{n-2}\alpha^{n-1}} \lambda_1 + \frac{1}{q^{n-1}\alpha^n} \lambda_1\right].$$

Because the expressions between brackets are the equations for $\underset{\sim}{\lambda}$ from Lemma 6.3, we find (6.10) for ν. ∎

LEMMA 6.5. *Consider an n-fraction as in* (6.4) *in which the coefficients are analytic function of a* (*some*) *complex variable*(*s*); q_0 *is as in Lemma* 6.3. *Then for each* $q > q_0$ *the n-sequences* $\{A_\nu^{(i)}/B_\nu\}_{\nu=1}^\infty$ $(i = 1,\dots, n)$ *converge to n analytic functions, uniformly on each compact subset of the domain* G_M *of the variable*(*s*), *defined by the following inequalities*

$$|a_1^{(i)}/b_1| \leqslant M,$$

$$|a_\nu^{(1)}/(b_{\nu-n} \cdots b_\nu)| \leqslant q/(q+1)^{\min(\nu,n+1)}, \tag{6.11}$$

$$|a_\nu^{(i)}/(b_{\nu-(n+1-i)} \cdots b_\nu)| \leqslant (q-1)/(q+1)^{\min(\nu,n+2-i)}$$

$$(i = 2,\dots, n) \qquad \text{for} \quad \nu \in \mathbb{N}\backslash\{1\}.$$

Proof. Because the differences $C_\nu^{(i)}D_{\nu-j} - C_{\nu-j}^{(i)}D_\nu$ are constructed in the same manner as the $\delta_{\nu,j}^{(i)}$ $(i, j = 1,\dots, n; \nu \in \mathbb{N})$ but with coefficients that are in absolute value less than those for the $\delta_{\nu,j}^{(i)}$ (see the proof of Lemma 6.2), we get

$$|C_\nu^{(i)}D_{\nu-j} - C_{\nu-j}^{(i)}D_\nu| \leqslant \delta_{\nu,j}^{(i)} \qquad (i, j = 1,\dots, n; \nu \in \mathbb{N}).$$

Now apply Lemma 6.2 with $p_\nu = q + 1$ $(\nu \in \mathbb{N})$ and Lemma 6.4 for $q > q_0$.

$$\left|\frac{A_\nu^{(i)}}{B_\nu} - \frac{A_{\nu-1}^{(i)}}{B_{\nu-1}}\right| \leqslant KN(\alpha_0 q)^\nu \Big/ \left(\sum_{\mu=0}^{\nu} k_\mu q^{\nu-\mu} \sum_{\mu=0}^{\nu-1} k_\mu q^{\nu-\mu-1}\right)$$
$$\leqslant KN(\alpha_0 q)^\nu/(c_1 q^\nu \cdot c_2 q^{\nu-1}) \qquad (c_1, c_2 \text{ not depending on } \nu)$$
$$= \beta(\alpha_0/q)^\nu \qquad (i = 1,\dots, n; \nu \in \mathbb{N}\backslash\{1\}) \text{ with } \beta = KN\alpha_0/(c_1c_2).$$

Together with $|A_1^{(i)}/B_1| = |a_1^{(i)}/b_1| \leqslant M$ $(i = 1,\dots, n)$, this shows that the n series $A_1^{(i)}/B_1 + \sum_{\nu=2}^\infty (A_\nu^{(i)}/B_\nu - A_{\nu-1}^{(i)}/B_{\nu-1})$ are uniformly convergent on $G_M(\alpha_0 < q)$; because $B_\nu = b_1 \cdots b_\nu D_\nu/(q+1)^\nu$ has no zeros on G_M (Lemma 6.2(b)), the assertion follows. ∎

LEMMA 6.6. *Let* $\beta_1, \dots, \beta_n$ *be n real numbers with*

$$0 < \beta_n < 1, 0 < \beta_{j-1} < \beta_j(1 - \beta_n) \qquad (j = 2,\dots, n) \tag{6.12}$$

and let an n-fraction of the form (6.4) with $b_\nu = 1$ $(\nu \in \mathbb{N})$ be given. $S_\nu : \mathbb{C}^n \to \mathbb{C}^n$ is the map defined by $S_\nu(x_1, \dots, x_n) = (S_\nu^{(1)},\dots, S_\nu^{(n)})$, where $S_\nu^{(i)}$ $(i = 1,\dots, n)$ follow from (1.7) and (1.8); $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid |x_i| \leqslant \beta_i \ (i = 1,\dots, n)\}$.

Furthermore let $a_\nu^{(i)}$ ($i = 1,..., n$; $\nu \in \mathbb{N}$) satisfy

$$|a_\nu^{(1)}| \leqslant \beta_1(1 - \beta_n), \qquad |a_\nu^{(i)}| \leqslant \beta_i(1 - \beta_n) - \beta_{i-1}$$
$$(i = 2,..., n) \qquad \text{for} \quad \nu \in \mathbb{N}. \qquad (6.13)$$

Then $S_\nu(V) \subset V$ ($\nu \in \mathbb{N}$).

Proof. By induction on ν from (1.6), (1.7), (1.8), and (6.12). ▮

LEMMA 6.7. (Stieltjes–Vitali) *Let $\{f_\nu(x)\}_{\nu=1}^\infty$ be a sequence of functions analytic on a domain D with*

(a) $|f_\nu(x)| \leqslant M(\nu \in \mathbb{N}, x \in D)$

(b) *$f_\nu(x)$ tends to a limit as $\nu \to \infty$ on a subset of D that contains infinitely many points and at least one limitpoint inside D.*

Then $\{f_\nu(x)\}_{\nu=1}^\infty$ converges uniformly in x on each compact subset in the interior of D, the limit function therefore being analytic on D.

Proof. See Stieltjes [10] and Titchmarsh [13]. ▮

Proof of Theorem 5.1. It is obvious from Definition 1.1. that we can restrict ourselves to C-n-fractions from Definition 3.1, satisfying (5.1) with $b_0^{(i)} = 0$ ($i = 1,..., n$); the approximants are denoted by $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{i=1}^n$ ($\nu \in \mathbb{N}_o$). With the coefficients of this n-fraction we form another one as in Lemma 6.2 (formula (6.4)), with $a_\nu^{(i)} = a_{i,\nu}x^{n+1-i}$ ($i = 1,..., n$), $b_\nu = 1$ for $\nu \in \mathbb{N}$; denote the approximants by $\{C_\nu^{(i)}(x)/D_\nu(x)\}_{i=1}^n$ ($\nu \in \mathbb{N}_o$). From Theorem 3.1 we derive, using $\rho_{-j} = x$ ($j = 0, 1,..., n - 1$), $\rho_\nu = 1$ ($\nu \in \mathbb{N}$)

$$C_\nu^{(i)}(x) = x^{n+1-i}A_\nu^{(i)}(x) \quad (i = 1,..., n), \qquad D_\nu(x) = B_\nu(x) \quad \text{for } \nu \in \mathbb{N}_0. \quad (6.14)$$

Now take $\beta_n = 2^{-1}, \beta_{n-i} = 2^{-i-1} - 2^{-i+1}\rho - 2^{-i+2}\rho^2 - \cdots - \rho^i (i = 1,..., n - 1)$ in (6.12) with ρ defined by (5.2). Because in that case $\rho = (2^{-1}\beta_1)^{1/n} = (2^{-1}\beta_i - \beta_{i-1})^{1/(n+1-i)}$ ($i = 2,..., n$), the conditions are satisfied for $x \in \mathfrak{D}$, $\mathfrak{D}$ defined by (5.4). The approximants $C_\nu^{(i)}(x)/D_\nu(x)$ ($i = 1,..., n$; $\nu \in \mathbb{N}$) therefore are bounded rational functions of x on $\mathfrak{D}$ and thus analytic. Finally apply Lemma 6.7 using the fact that there exists a small circular domain around $x = 0$ on which the n-fraction converges as can be seen by Lemma 6.5. This leads to uniform convergence of $\{C_\nu^{(i)}(x)/D_\nu(x)\}_{\nu=1}^\infty$ ($i = 1,..., n$) on compact subsets of $\mathfrak{D}$ and therefore to the same for the sequences $\{A_\nu^{(i)}(x)/B_\nu(x)\}_{\nu=1}^\infty$ ($i = 1,..., n$) on compact subsets of $\mathfrak{D}\backslash\{0\}$. Applying Lemma 6.5 to the n-fraction with approximants $A_\nu^{(i)}(x)/B_\nu(x)$ directly, we derive the existence of a (maybe very small) $\delta > 0$, such that the sequences converge uniformly on $|x| < \delta$. Combining two overlapping regions of uniform convergence, we are led to the assertion of the theorem using analytic continuation. ▮

Proof of Theorem 5.2. Apply Theorem 2.1 (formulas (2.2) and (2.3)) and Theorem 5.1 on $(1-\epsilon)\mathfrak{D} = \{y \in \mathbb{C} \mid y = (1-\epsilon)x, x \in \mathfrak{D}\}$ $(\epsilon \downarrow 0)$, where λ in (2.2) follows from

$$\min_i(\eta_i^{-1/(n+1-i)}) < (1-\epsilon)\min_i(a_i^{-1/(n+1-i)})$$

with $\eta_i = \sup_{\nu \geqslant \lambda} | a_{i,\nu} |$ and a_i as before $(i = 1,..., n)$.

The poles come in when piecing the two parts of the n-fraction together. It is obvious from Corollary 1.1 that a pole of one of the limit functions is a zero x_0 of the denominator of the rational functions

$$(\xi_\nu^{(n)} A_{\nu-1}^{(i)} + \cdots + \xi_\nu^{(1)} A_{\nu-n}^{(i)} + a_{1,\nu}x^n A_{\nu-n-1}^{(i)})/$$
$$(\xi_\nu^{(n)} B_{\nu-1} + \cdots + \xi_\nu^{(1)} B_{\nu-n} + a_{1,\nu}x^n B_{\nu-n-1})$$

$(i = 1,..., n)$ for some $\nu \geqslant \nu_0$ (the $\xi_\nu^{(i)}$ as in Section 2; $\xi_\nu^{(1)}(0) = \cdots = \xi_\nu^{(n-1)}(0) = 0$, $\xi_\nu^{(n)}(0) = 1$ for $\nu \in \mathbb{N}$; the denominator does not vanish identically because it has the value 1 at $x = 0$). If there is a value of j for which the numerator is different from zero for $x = x_0$, we can apply Theorem 2.4. If all numerators vanish for $x = x_0$ (x_0 clearly satisfies $x_0 \neq 0$), we would have a system of homogeneous linear equations

$$\mathfrak{A}_{\nu-1}(\xi_\nu^{(n)}(x_0),..., \xi_\nu^{(1)}(x_0), a_{1,\nu}x_0^{\,n})^T = \underline{0}$$

which has a nontrivial solution $(a_{1,\nu}x_0^{\,n} \neq 0)$, contradictory to the fact that $\det \mathfrak{A}_{\nu-1} \neq 0$ as follows from the restrictions on $a_{1,\nu} (\nu \in \mathbb{N})$ and formula (1.10). ∎

Proof of Theorem 5.3. The main part follows from Theorem 5.1 while the linear independence follows by Theorem 4.6 once we have proved $f^{(i)} \equiv g^{(i)}$ on $\mathfrak{D}$ $(i = 1,..., n)$. This follows from the so-called "Weierstrasschen Doppelreihensatz" (see Perron [7, page 147]), applied to each of the n series of approximants separately. Using analytic continuation and the fact that the convergence is uniformly in x on compact subsets of $\mathfrak{D}$, it is obvious that the singularities of the $f^{(i)}$ $(i = 1,..., n)$, if there are any, cannot lie in the interior of $\mathfrak{D}$. ∎

Proof of Theorem 5.4. Follows from Theorem 5.2. In the same manner as in the proof of Theorem 5.3 we can show that the poles of $f^{(1)},..., f^{(n)}$ cannot lie in the interior of $\mathfrak{D}$ and that $f^{(i)} \equiv g^{(i)}$ on $\mathfrak{D}$ $(i = 1,..., n)$, whence the linear independence of the g's by Theorem 4.6. ∎

Proof of Example 5.2. Define the quantities $x_\nu^{(i)}$ ($i = 1,\dots, n$; $\nu \in \mathbb{N}$) by

$$\begin{aligned} x_\nu^{(n-j)} &= \sum_{k=j}^{n} \frac{B_k(\alpha_1 + \nu - j,\dots, \alpha_n + \nu - j)}{\prod_{i=1}^n (\alpha_i + \nu - j)_{k+1}} x^k \\ &\quad \times {}_0F_n(\alpha_1 + k + \nu - j + 1,\dots, \alpha_n + k + \nu - j + 1; x) \qquad (j = 1,\dots, n), \quad (6.15) \\ x_\nu^{(n)} &= {}_0F_n(\alpha_1 + \nu,\dots, \alpha_n + \nu; x). \end{aligned}$$

Substituting appropriate values for $\beta_1,\dots, \beta_n$ in

$$\begin{aligned} &{}_0F_n(\beta_1,\dots, \beta_n ; x) - {}_0F_n(\beta_1 + 1,\dots, \beta_n + 1; x) \\ &= \sum_{j=0}^{\infty} x^j \left\{\prod_{i=1}^{n} (\beta_i + j) - \prod_{i=1}^{n} \beta_i\right\} \Big/ \left\{ j! \prod_{i=1}^{n} (\beta_i)_{j+1} \right\} \\ &= \sum_{k=1}^{n} \frac{B_k(\beta_1,\dots, \beta_n)}{\prod_{i=1}^n (\beta_i)_{k+1}} \sum_{j=k}^{\infty} (j-k+1)_k\, x^j \Big/ \left\{ j! \prod_{i=1}^{n} (\beta_i + k + 1)_{j-k} \right\} \\ &= \sum_{k=1}^{n} \frac{B_k(\beta_1,\dots, \beta_n)}{\prod_{i=1}^n (\beta_i)_{k+1}} x^k\, {}_0F_n(\beta_1 + k + 1,\dots, \beta_n + k + 1; x), \end{aligned}$$

we find that the quantities from (6.15) satisfy

$$\begin{aligned} x_\nu^{(n)} &= x_{\nu+1}^{(n)} + x_{\nu+1}^{(n-1)}, \\ x_\nu^{(n-j)} &= \left\{ B_j(\alpha_1 + \nu - j,\dots, \alpha_n + \nu - j)\, x^j \Big/ \prod_{i=1}^{n} (\alpha_i + \nu - j)_{j+1} \right\} x_{\nu+1}^{(n)} \\ &\quad + x_{\nu+1}^{(n-j-1)} \qquad (j = 1,\dots, n-1), \qquad (6.16) \\ x_\nu^{(0)} &= \left\{ x^n \Big/ \prod_{i=1}^{n} (\alpha_i + \nu - n)_{n+1} \right\} x_{\nu+1}^{(n)}. \end{aligned}$$

Then the definition $f_\mu^{(i)} = x_\mu^{(i)}/x_{\mu+1}^{(n)}$ ($i = 1,\dots, n$; $\mu \in N_0$) leads to a non-terminating set of equations like (1.4) (or as in Theorem 3.3), with

$$b_0^{(n-j)} = \frac{B_j(\alpha_1 - j,\dots, \alpha_n - j)}{\prod_{i=1}^n (\alpha_i - j)_{j+1}} x^j \qquad (j = 1,\dots, n-1), \qquad b_0^{(n)} = 1,$$

and for $\mu \in \mathbb{N}$

$$a_\mu^{(n+1-j)} = \frac{B_j(\alpha_1 + \mu - j,\dots, \alpha_n + \mu - j)}{\prod_{i=1}^n (\alpha_i + \mu - j)_{j+1}} x^j \qquad (j = 1,\dots, n), \qquad b_\mu = 1.$$

Comparing this with the construction in Theorem 3.3, we see that the starting functions are $f_0^{(1)},..., f_0^{(n)}$, given by

$$f_0^{(n-j)}(x) = \sum_{k=j}^{n} \frac{B_k(\alpha_1 - j,..., \alpha_n - j)}{\prod_{i=1}^{n} (\alpha_i - j)_{k+1}} x^k$$
$$\times \frac{{}_0F_n(\alpha_1 + k - j + 1,..., \alpha_n + k - j + 1; x)}{{}_0F_n(\alpha_1 + 1,..., \alpha_n + 1; x)} \qquad (j = 1,..., n - 1),$$
$$f_0^{(n)}(x) = \frac{{}_0F_n(\alpha_1 ,..., \alpha_n ; x)}{{}_0F_n(\alpha_1 + 1,..., \alpha_n + 1; x)} .$$

Applying the construction of Theorem 3.3 to the functions $x^{-j} f_0^{(n-j)}(x)$ $(j = 0, 1,..., n - 1)$, we find that the functions in (5.6) indeed have a regular C-n-fraction with coefficients given by (5.7); this is also obvious from Theorem 1.3 with multiplicators $\rho_{-j} = x^{-1}$ $(j = 1,..., n - 1)$, $\rho_\nu = 1$ $(\nu \in \mathbb{N}_0)$. The matter of convergence is easily settled, because we can give estimates for $B_k(\beta_1 ,..., \beta_n)$ $(k = 1,..., n)$ using the elementary symmetric functions in $\beta_1 ,..., \beta_n$ and Stirling numbers of the second kind. There exists a constant M, *only depending on n*, such that we get with $\beta = \max\{1, \beta_1 ,..., \beta_n\}$

$$B_k(\beta_1 ,..., \beta_n) \leqslant M\beta^{n-k} \qquad (k = 1,..., n).$$

Inserting this bound in (5.7), we have with $\alpha = \max\{\alpha_1 ,..., \alpha_n\}$

$$| a_{n+1-k,\nu} | \leqslant M(\alpha + \nu - k)^{n-k}/(\nu - k)^{n(k+1)} \quad (k = 1,..., n; \nu \geqslant n + 1),$$

after which application of Theorem 5.2 leads to the desired result. ∎

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